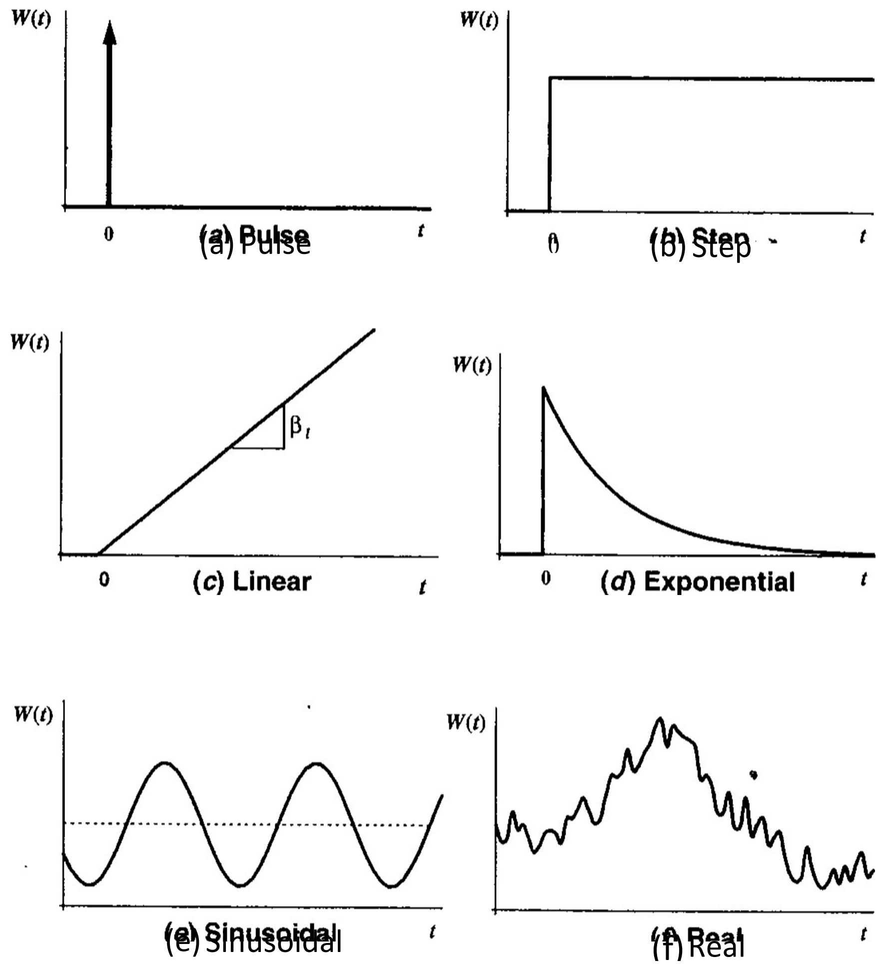
Lecture 4:
Particular Solutions to Selected Forcing Functions

- To develop solutions (i.e., "particular) for specific loading characteristics, such as:
  - Impulse load
  - Step load
  - Linear load
  - Exponential load
  - Sinusoidal load
- To find analytical solutions and any associated "shape parameters" with each unique solution

Usefulness

- Approximations needed to support complex situations for decision-making
- Idealizations are important to predict unknown future loads
- Idealized loading functions allow gaining a better understanding of how a model works.

Loadings Functions W(t) Versus Time



Forcing Function and Response

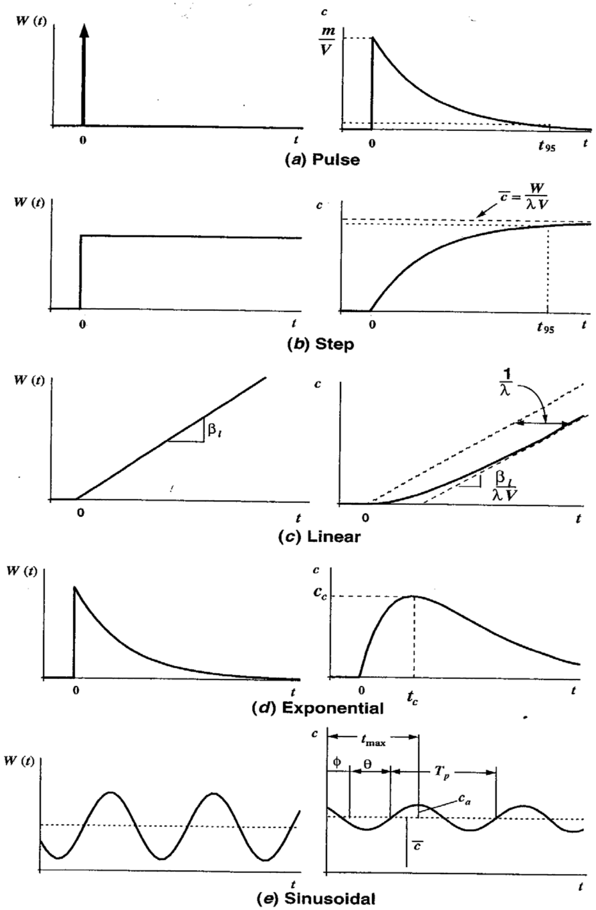


FIGURE 4.7
A summary of the solutions developed to this point for a substance reacting with first-order kinetics. Note that the shape parameters are shown on the concentration response plots.

Impulse Loading (Spill)

- Pollutant discharge over a relatively very short time period (i.e., seconds, minutes).
- *Dirac delta* function or impulse function, $\delta(t)$ in time^{-1} units (is 0 at t=0 and has a unit area over time), is the forcing function:

$$\mathbf{dc/dt + \lambda c = W(t)/V = m\delta(t)/V}$$

whose solution yields:

$$\mathbf{c = (m/V)e^{-\lambda t} = c_o e^{-\lambda t}}$$

Impulse Loading (spill)

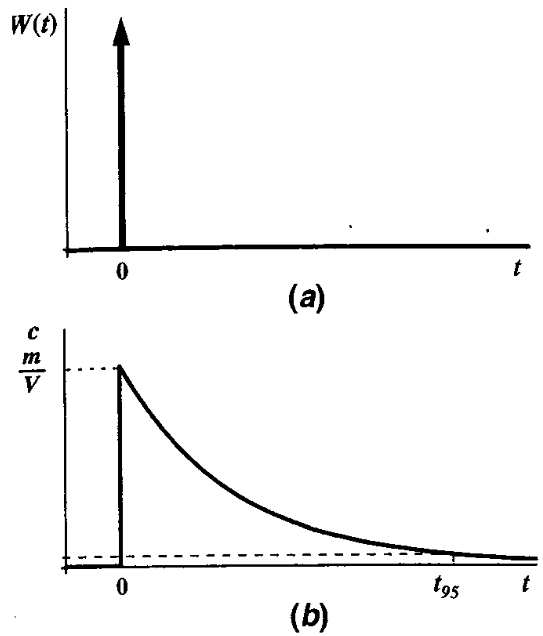


FIGURE 4.2
Plot of (*a*) loading and (*b*) response for impulse loading.

Step Loading

- New continuous source
- Forcing function is a step input ($W(t) = 0, t<0$ & $W(t) = W, t \leq 0$); W in Mt^{-1})

$$\mathbf{c = (W / \lambda V)(1 - e^{-\lambda t})}$$

at steady state,

$$\mathbf{c_{ss} = W / \lambda V}$$

Step Loading (continuous source)

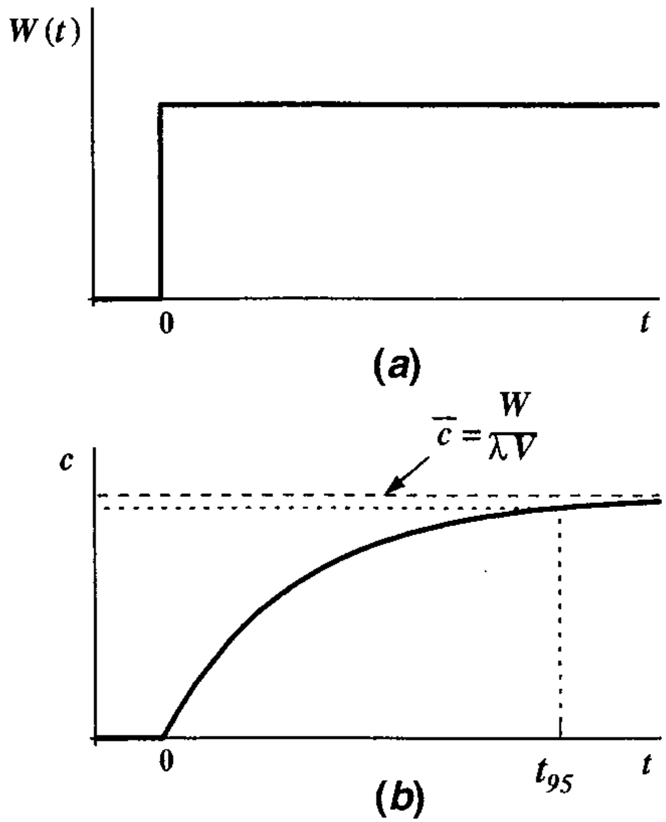


FIGURE 4.3
Plot of (*a*) loading and (*b*) response for step loading.

EXAMPLE 4.1. STEP LOADING. At time zero, a sewage treatment plant began to discharge 10 MGD of wastewater with a concentration of 200 mg L^{-1} to a small detention basin (volume $= 20 \times 10^4$ m^3). If the sewage decays at a rate of 0.1 d^{-1}, compute the concentration in the system during the first 2 wk of operation. Also determine the shape parameters to assess the ultimate effect of the plant.

Solution: The flow must be converted to the proper units:

$$10 \text{ MGD} \frac{1 \text{ m}^3 \text{ s}^{-1}}{22.8245 \text{ MGD}} \left(\frac{86{,}400 \text{ s}}{\text{d}} \right) = 37{,}854 \text{ m}^3 \text{ d}^{-1}$$

The eigenvalue can be determined as

$$\lambda = \frac{37{,}854}{20 \times 10^4} + 0.1 = 0.28927 \text{ d}^{-1}$$

Therefore the concentration can be computed with Eq. 4.10 as

$$c = \frac{W}{\lambda V}(1 - e^{-\lambda t}) = \frac{200(37{,}854)}{0.28927(20 \times 10^4)}(1 - e^{-0.28927t}) = 131(1 - e^{-0.28927t})$$

The results for the first 2 wk are

t (d)	0	2	4	6	8	10	12	14
c (mg L^{-1})	0	57.48	89.72	107.79	117.92	123.61	126.79	128.58

These values can also be displayed graphically as

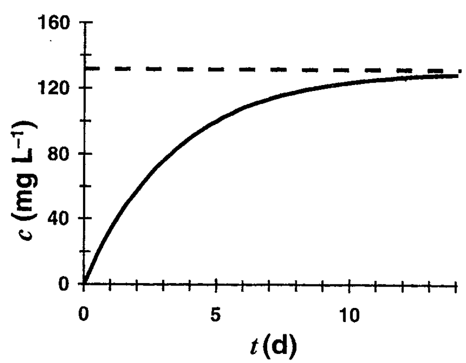


FIGURE E4.1

The shape parameters for this case are an ultimate steady-state concentration of 131 mg L^{-1}, of which 95% will be attained in $3/0.28927 \cong 10.4$ d.

Linear ("Ramp") Loading

Waste loading increases or decreases exponentially, as follows:

$$\mathbf{W(t) \quad = \quad \pm\beta_1\, t} \text{ (with units of } Mt^{-2}\text{)}$$

where β_1 = rate of change or slope of the trend (if $+\beta$, population growth effects can be simulated)

The analytical solution of our DE is:

$$\mathbf{c \quad = \quad [\pm\beta_1/(\lambda^2 V)](\lambda t - 1 + e^{-\lambda t})}$$

Linear ("Ramp") Loading

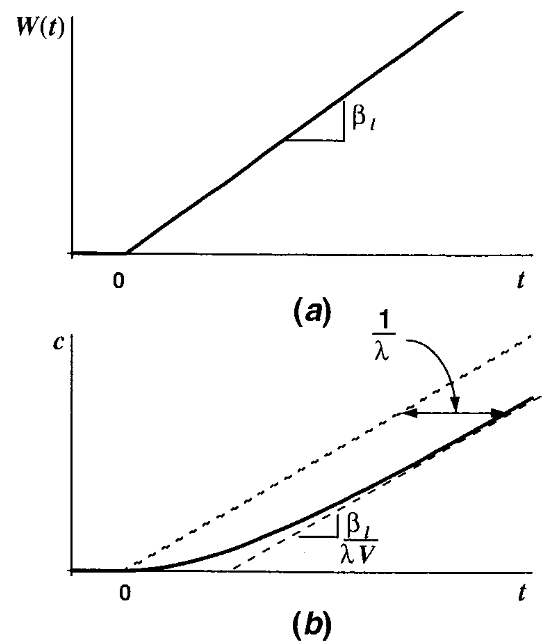


FIGURE 4.4
Plot of (*a*) loading and (*b*) response for a linearly increasing loading.

Exponential Loading

Loading is $W(t) = W_e\, e^{\pm\beta e\, t}$, which yields:

$$c = [(W_e)/(V(\lambda \pm \beta_e))]\,[e^{\pm\beta e\, t} - e^{-\lambda t}]$$

where W_e = a parameter that denotes the value at $t = 0$ (Mt^{-1})

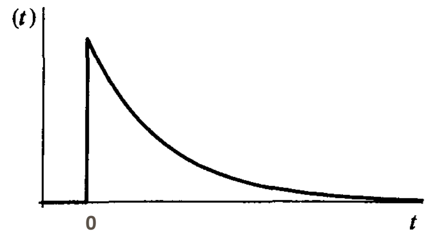
β_e = growth or decay rate of loading (t^{-1})

At critical condition, where c is a maximum, from $dc/dt = 0$:

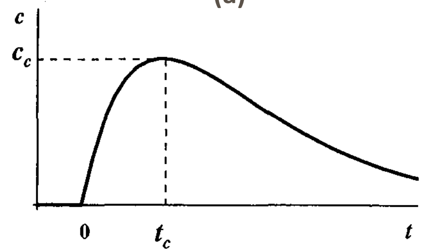
$$t_c = \ln(\beta_e/\lambda)/(\beta_e - \lambda)$$

$$c_c = (W_e/\lambda V)(\beta_e/\lambda)^{\beta e/(\lambda - \beta e)}$$

Exponential Loading



(a)



(b)

EXAMPLE 4.2. EXPONENTIAL FORCING FUNCTION. The following series of first-order reactions takes place in a batch reactor:

$$A \xrightarrow{k_1} B \xrightarrow{k_2}$$

Mass balance equations for these reactions can be written as

$$\frac{dc_A}{dt} = -k_1 c_A$$

and

$$\frac{dc_B}{dt} = k_1 c_A - k_2 c_B$$

Suppose that an experiment is conducted where $c_{A0} = 20$ and $c_{B0} = 0$ mg L^{-1}. If $k_1 = 0.1$ and $k_2 = 0.2$ d^{-1}, compute the concentration of reactant B as a function of time. Also, determine its shape parameters.

Solution: The concentration of reactant A can be determined by integrating the first differential equation to give

$$c_A = c_{A0} e^{-k_1 t}$$

This result can be substituted into the second differential equation to yield

$$\frac{dc_B}{dt} + k_2 c_B = k_1 c_{A0} e^{-k_1 t}$$

Thus the mass balance is now in the form of a first-order differential equation with an exponential forcing function. The solution is

$$c_B = \frac{k_1 c_{A0}}{(k_2 - k_1)} \left(e^{-k_1 t} - e^{-k_2 t}\right)$$

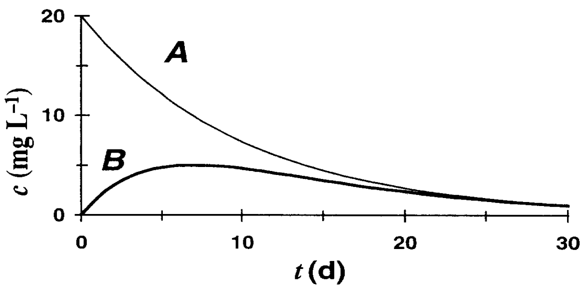


FIGURE E4.2

or substituting the parameter values,

$$c_B = 20\left(e^{-0.1t} - e^{-0.2t}\right)$$

The plot of this solution, along with the solution for c_A, is shown in Fig. E4.2.

The shape parameters for this case can be computed as

$$c_c = \frac{0.1(20)}{0.2}\left(\frac{0.1}{0.2}\right)^{\frac{0.1}{0.2-0.1}} = 5 \text{ mg L}^{-1}$$

which will occur at

$$t_c = \frac{\ln(0.1/0.2)}{0.1 - 0.2} = 6.93 \text{ d}$$

These values are consistent with the graph.

Sinusoidal Loading

- A simple periodic input is represented as $W(t) = \hat{W} = W_a \sin(\omega t - \theta)$
- A solution, when $\theta = 0$, inflow = outflow, and $W(t) = Qc_{a,in}\sin(\omega t)$, is

$$c = c_{a,in} A(\omega) \sin[\omega t - \phi(\omega)]$$

Sinusoidal Loading

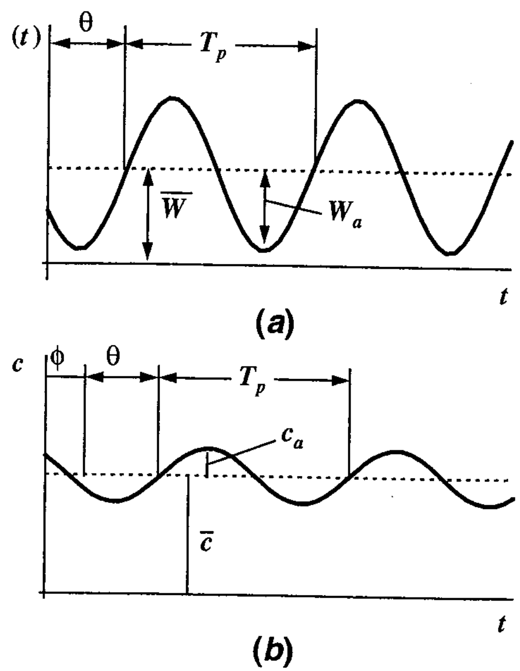


FIGURE 4.6
Plot of (*a*) loading and (*b*) response for the sinusoidal loading function. Note that a constant input is also shown in this illustration.

EXAMPLE 4.3. SINUSOIDAL FORCING FUNCTION. A completely mixed lake receives a conservative substance. Water inflow and outflow are equal, and the inflow concentration varies sinusoidally as

$$c_{\text{in}} = \bar{c}_{\text{in}} + c_{a,\text{in}} \sin(\omega t)$$

where $\bar{c}_{\text{in}}$ = average inflow concentration
$c_{a,\text{in}}$ = amplitude of the inflow concentration
ω = angular frequency ($= 2\pi/T_p$), in which T_p = period of the oscillation

If the lake has a volume of 2.5×10^6 m^3 and inflow = outflow = 9×10^6 m^3 yr^{-1}, determine its sensitivity to the sinusoidal component of the loading if the period of the oscillation is (*a*) 10 yr, (*b*) 1 yr, or (*c*) 0.1 yr.

Solution: The eigenvalue can be computed as

$$\lambda = \frac{Q}{V} = \frac{9 \times 10^6}{2.5 \times 10^6} = 3.6 \text{ yr}^{-1}$$

For a period of 10 yr, $\omega = 2\pi/10 = 0.628$ yr^{-1} and

$$\phi(0.628) = \tan^{-1}\left(\frac{0.628}{3.6}\right) = 0.1727 \text{ radian}\left(\frac{10 \text{ yr}}{2\pi \text{ radians}}\right) = 0.275 \text{ yr (100 d)}$$

and $$A(0.628) = \frac{3.6}{\sqrt{3.6^2 + 0.628^2}} = 0.985$$

Therefore the solution is almost identical to the forcing function. The amplitude is diminished only 1.5% and the phase shift is a mere 100 d (compared to the period of 10 yr). However, as shown by the following table, this correspondence breaks down as the frequency of the forcing function increases:

Period (yr)	f (cycles yr^{-1})	$A(\omega)$	$\phi(\omega)$ (d)	$[\phi(\omega)/T_p] \times 100\%$
10	0.1	0.985	100	2.8%
1	1	0.573	61	16.7%
0.1	10	0.057	8.8	24.1%

As the frequency increases, the solution exhibits two effects. First, the amplitude diminishes and approaches zero. Second, as indicated by the normalized phase shift, the solution increasingly lags the forcing function. In essence both effects reflect the fact that the system is too sluggish (as manifested by its eigenvalue, λ) to "keep up" with the forcing function.

It should be noted that there is a formal way of presenting the information developed in this example. That is, both the amplitude and phase characteristics can be plotted versus frequency in what are called ***Bode diagrams***. These are conventionally constructed by plotting either the amplitude or the phase characteristic versus the angular frequency (ω). As shown in Fig. E4.3, we have chosen to plot the characteristics versus the period. We have also plotted the phase shift in degrees. In the present context we believe that both modifications make the plots somewhat easier to interpret.

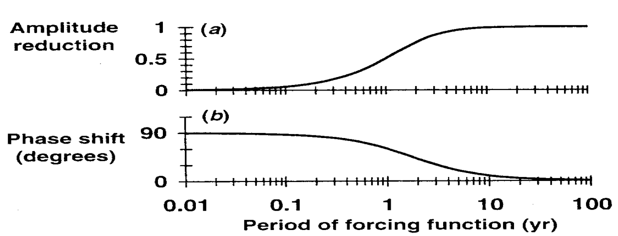


FIGURE E4.3

Linearity and Time Shifts

- Linearity:
  - $c = c_g + \Sigma a_i c_{pi}$
- Time shifts:
  - $c(t)$ & $c(t-a)$, where a is the shift

EXAMPLE 4.4. LINEAR AND EXPONENTIAL LOADINGS. O'Connor and Mueller (1970) used linear and exponential forcing functions to characterize the loadings of a conservative substance, chloride, to Lake Michigan. For example they characterized chloride loadings due to road salt by the linear model

$$W(t) = 0 \qquad t < 1930$$

$$W(t) = 13.2 \times 10^9(t - 1930) \qquad 1930 \le t \le 1960$$

where $W(t)$ has units of g yr^{-1}. They used an exponential model to characterize other sources of salt that were correlated with population growth in the basin (for example municipal and industrial sources),

$$W(t) = 0 \qquad t < 1900$$

$$W(t) = 229 \times 10^9 e^{0.015(t-1900)} \qquad 1900 \le t \le 1960$$

Finally they considered that the lake had a background chloride concentration of 3 mg L^{-1}.

According to O'Connor and Mueller, Lake Michigan had the following average characteristics for the period from 1900 to 1960: outflow = 49.1×10^9 m^3 yr^{-1} and volume = 4880×10^9 m^3. Calculate the chloride concentration in Lake Michigan from 1900 through 1960.

Solution: Because chloride is a conservative substance, the eigenvalue is simply the reciprocal of the residence time:

$$\lambda = \frac{Q}{V} = \frac{49.1 \times 10^9}{4880 \times 10^9} = 0.01\ \text{yr}^{-1}$$

Therefore the solution is

From 1900 to 1930:

$$c = 3 + \frac{229 \times 10^9}{4880 \times 10^9(0.01 + 0.015)}(e^{+0.015(t-1900)} - e^{-0.01(t-1900)})$$

From 1930 to 1960:

$$c = 3 + \frac{229 \times 10^9}{4880 \times 10^9(0.01 + 0.015)}(e^{+0.015(t-1900)} - e^{-0.01(t-1900)})$$

$$- \frac{13.2 \times 10^9}{(0.01)^2 4880 \times 10^9}[1 - e^{-0.01(t-1930)} - 0.01(t - 1930)]$$

Note that initial conditions are not included explicitly because they are reflected in the constant background level of 3 mg L^{-1}. The results, along with data, are depicted in Fig. E4.4.

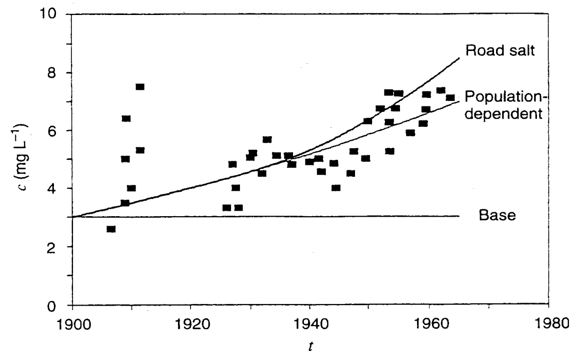


FIGURE E4.4

EXAMPLE 4.5. HALF-SINUSOID FORCING FUNCTION. The half-sinusoid (Fig. 4.8*a*) can be employed in a number of water-quality-modeling contexts. For example it has been used successfully to simulate the affect of diurnal variations of plant photosynthesis on stream dissolved-oxygen levels.

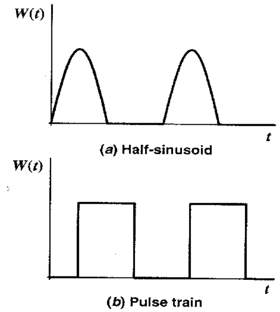


FIGURE 4.8
Two commonly used periodic inputs.
(*a*) Half-sinusoid; (*b*) pulse train.

Another example relates to the design of a detention basin. Suppose that during the summer a company discharges a small flow of a highly concentrated contaminant into a tributary of a larger river. The company discharges from 6:00 P.M. until 6:00 A.M. Measurements at the mouth of the tributary indicate that a half-sine wave provides an adequate approximation of the concentration time series over the period

$$c_{\text{in}}(t) = c_a \sin(\omega t) \qquad 0 \le t \le \frac{T_p}{2}$$

$$c_{\text{in}}(t) = 0 \qquad \frac{T_p}{2} \le t \le T_p$$

where $c_{\text{in}}(t)$ = time series of the inflow concentration at the tributary mouth (μg L^{-1}) and c_a = amplitude of the half-sinusoid (μg L^{-1}). Note that time zero is assumed to be at 6:00 P.M.

It is proposed that the company build a detention basin at the tributary mouth to moderate the effect of the discharge on the river. The flow in the tributary (including the discharge) can be adequately characterized by a constant level of $Q = 1 \times 10^6$ m^3 d^{-1}. In addition the detention basin's volume is 1×10^6 m^3 and the amplitude of the concentration is 10 μg L^{-1}. If no losses occur, determine the long-term concentration response using the well-mixed-lake model (Eq. 4.1) along with a Fourier-series approximation of the half-sinusoidal loading up to the second harmonic.

Solution: The mass balance for the detention basin can be written as

$$\frac{dc}{dt} + \lambda c = \lambda c_{\text{in}}(t)$$

where the eigenvalue for this example is $Q/V = 1$. The coefficients for the Fourier-series approximation of the half-sinusoid can be evaluated, as in

$$a_0 = c_a \frac{1}{1} \int_0^{1/2} \sin(2\pi t)\, dt = c_a \frac{1}{\pi}$$

$$a_1 = c_a \frac{2}{1} \int_0^{1/2} \sin(2\pi t) \cos(2\pi t)\, dt = 0$$

$$b_1 = c_a \frac{2}{1} \int_0^{1/2} \cos(2\pi t) \sin(2\pi t)\, dt = c_a \left(\frac{1}{2}\right)$$

$$a_2 = c_a \frac{2}{1} \int_0^{1/2} \sin(2\pi t) \cos(4\pi t)\, dt = c_a \left(-\frac{2}{3\pi}\right)$$

$$b_2 = c_a \frac{2}{1} \int_0^{1/2} \sin(2\pi t) \sin(4\pi t)\, dt = 0$$

Therefore the loading function can be approximated by (with $c_a = 10$)

$$c_{\text{in}}(t) = \frac{10}{\pi} + 5\sin(2\pi t) - \frac{20}{3\pi} \cos(4\pi t)$$

Note that the last term can be represented as a sine with a phase shift of $\pi/2$,

$$\sin\left(4\pi t + \frac{\pi}{2}\right) = \cos(4\pi t)$$

Therefore the loading function becomes

$$c_{\text{in}}(t) = \frac{10}{\pi} + 5\sin(2\pi t) - \frac{20}{3\pi}\sin\left(4\pi t + \frac{\pi}{2}\right)$$

This approximation is displayed in Fig. 4.9*a*. Notice that although it is not perfect (there are even some slightly negative values), the series provides a reasonable approximation of the half-sinusoid.

To evaluate concentration, the approximation is substituted into the mass balance to give

$$\frac{dc}{dt} + \lambda c = \frac{10\lambda}{\pi} + 5\lambda\sin(2\pi t) - \frac{20\lambda}{3\pi}\sin\left(4\pi t + \frac{\pi}{2}\right)$$

The response can be determined as

$$c = \frac{10}{\pi} + \frac{5\lambda}{\sqrt{\lambda^2 + (2\pi)^2}}\sin(2\pi t - \theta_1) - \frac{20\lambda}{3\pi\sqrt{\lambda^2 + (4\pi)^2}}\sin\left(4\pi t + \frac{\pi}{2} - \theta_2\right)$$

where

$$\theta_1 = \tan^{-1}\left(\frac{2\pi}{1}\right) = 1.413 \text{ radians } (= 5.5 \text{ hr})$$

$$\theta_2 = \tan^{-1}\left(\frac{4\pi}{1}\right) = 1.491 \text{ radians } (= 5.7 \text{ hr})$$

The results, displayed in Fig. 4.9*b*, indicate that the detention pond has a definite effect on the concentration discharged to the river. Whereas the inflow concentration swings between 0 and 10 μg L^{-1}, the pond concentration moves between about 2.4 and 4.1 μg L^{-1}. Further, the peak occurs at about 4:00 A.M. rather than at midnight.

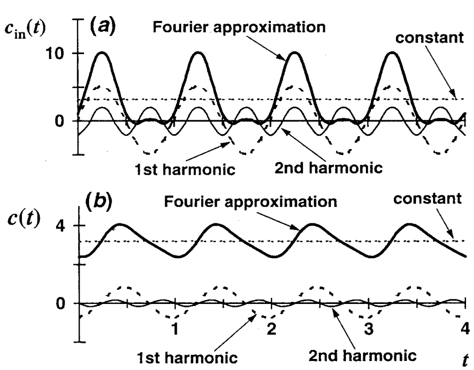
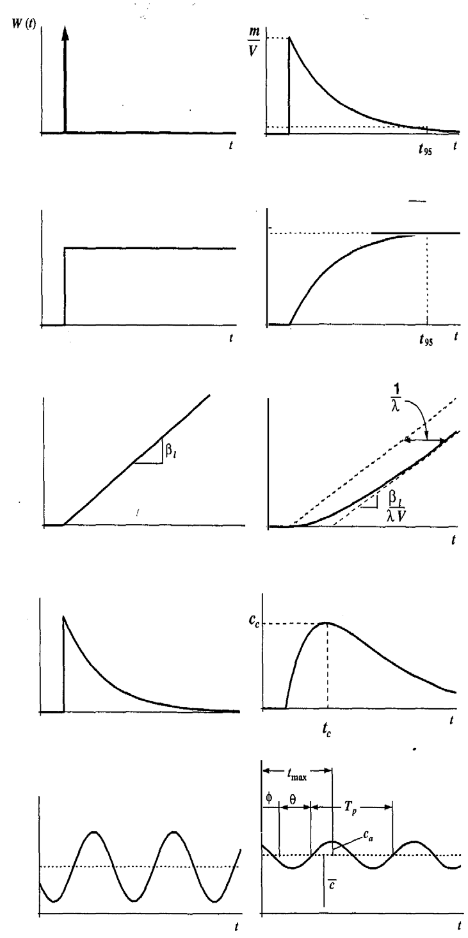


FIGURE 4.9
Loading and response for a detention basin. (*a*) Inflow concentration approximated by a half-sinusoid; (*b*) the resulting concentration in the basin.



W (t)
t
m
V
t
t_{95}
t
t
t_{95}
t
β_l
1
λ
β_l
λ V
t
c_c
t_c
t
t
t_{max}
φ
θ
T_p
c_a
$\bar{c}$
t

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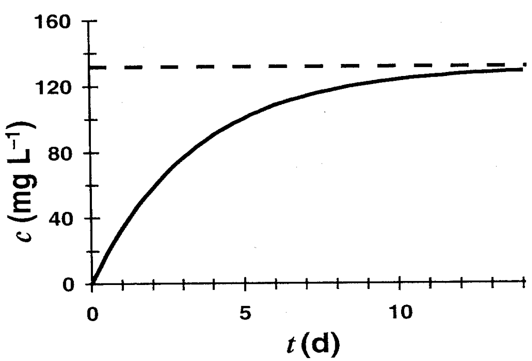

FIGURE E4.1

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