14.06 LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER

(1) Variable Coefficients $\quad y' + P(x)y = Q(x)$

Condition	Solution
$Q(x) = 0$	$y = C \exp\left[-\int P(x)\,dx\right]$
$Q(x) \neq 0$	$y = \exp\left[-\int P(x)\,dx\right] \times \left\{\int Q(x) \exp\left[\int P(x)\,dx\right] dx + C\right\}$

(2) Constant Coefficients $\quad (A, B, \alpha, \beta = \text{constants}) \quad y' + By = Q(x)$

Condition	Solution
$Q(x) = 0$	$y = Ce^{-Bx}$
$Q(x) = A$	$y = Ce^{-Bx} + \dfrac{A}{B}$
$Q(x) = Ax$	$y = Ce^{-Bx} + \dfrac{A}{B}\left(x - \dfrac{1}{B}\right)$
$Q(x) = Ax^2$	$y = Ce^{-Bx} + \dfrac{A}{B}\left(x^2 - \dfrac{2x}{B} + \dfrac{2}{B^2}\right)$
$Q(x) = Af(x)$	$y = Ce^{-Bx} + \dfrac{A}{B}\left[f(x) - \dfrac{f'(x)}{B} + \dfrac{f''(x)}{B^2} - \cdots\right]$
$Q(x) = Ae^{\alpha x}$	$y = Ce^{-Bx} + \dfrac{A}{\alpha + B}e^{\alpha x}$
$Q(x) = A \sin \beta x$	$y = Ce^{-Bx} + \dfrac{A(B \cos \beta x - \beta \sin \beta x)}{\beta^2 + B^2}$
$Q(x) = A \cos \beta x$	$y = Ce^{-Bx} + \dfrac{A(B \cos \beta x + \beta \sin \beta x)}{\beta^2 + B^2}$
$Q(x) = Ae^{\alpha x} \sin \beta x$	$y = Ce^{-Bx} - \dfrac{A[(\alpha + B) \sin \beta x + \beta \cos \beta x]}{\beta^2 + (\alpha + B)^2}$
$Q(x) = Ae^{\alpha x} \cos \beta x$	$y = Ce^{-Bx} + \dfrac{A[\beta \sin \beta x + (\alpha + B) \cos \beta x]}{\beta^2 + (\alpha + B)^2}$

(3) Bernoulli's Equation $\quad y' + P(x)y = Q(x)y^n$

Substitution:

$$y = z^{1/(1-n)} \qquad y' = \frac{1}{1-n} z^{n/(1-n)} z'$$

Reduced equation:

$$z' + (1-n)P(x)z = (1-n)Q(x)$$

Solution:

$$y^{1-n} = \exp\left[(n-1)\int P(x)\,dx\right]\left\{(1-n)\int Q(x) \exp\left[(1-n)\int P(x)\,dx\right] dx + C\right\}$$

SOURCE: J. J. Tuma
Engineering Mathematics Handbook, 2nd ed.
McGraw-Hill Book Company, 1979