GOVERNING EQUATION FOR UNSATURATED FLOW

The water table defines the boundary between the unsaturated and saturated zones and is defined by the surface at which the fluid pressure P is exactly atmospheric, or $P = 0$. Hence, the total hydraulic head $h = \psi + z$, where $\psi = P/\rho g$, the capillary pressure head.

Considerations of unsaturated flow include the solution of the governing equation of continuity and Darcy's law in an unsaturated porous media. The governing equation, originally derived by Richard in 1931, is based on substituting Darcy's law Eq. (9.6) into the unsaturated continuity equation,

$$-\left[\frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}\right] = \frac{\partial}{\partial t}\rho\theta \tag{9.7}$$

The resulting equation is

$$\frac{\partial\theta}{\partial t} = -\frac{\partial}{\partial z}\left[K(\theta)\frac{\partial\psi(\theta)}{\partial z}\right] - \frac{\partial K(\theta)}{\partial z} \tag{9.8}$$

where

θ = volumetric moisture content
z = distance below the surface [L]
θ = capillary suction (pressure) [L of water]
$K(\theta)$ = unsaturated hydraulic conductivity [L/T]

Equation (9.8) is called Richards (1931) equation and is a nonlinear partial differential equation that is quite difficult to solve. The Richards equation assumes that the presence of air can be ignored, that water is incompressible, and that the soil matrix is nondeformable. Both numerical and analytical solutions exist for certain special cases. The most difficult part of the procedure is determining the characteristic curves for a soil (Figure 9.2). The characteristic curves reduce to the fundamental hydraulic parameters K and n in the saturated zone and remain as functional relationships in the unsaturated zone.

A number of analytical and numerical solution techniques have been developed over the last 35 years for Eq. (9.8). Analytical approaches require greatly simplified conditions for boundaries, and are generally restricted to 1-D vertical systems. Early work by Philip and de Vries (1957) provided much of the physical and mathematical ground work for subsequent analyses. The analytical approaches are useful for deriving physically based expressions for infiltration rates in unsaturated soils. A more recent approach based on the method of characteristics has been applied to gravity-dominated flow in the unsaturated zone (Smith 1983; Charbeneau, 1984).