3.2 CONTROL VOLUME APPROACH FOR HYDROSYSTEMS

Hydrosystem processes transform the space and time distribution of water in hydrologic systems throughout the hydrologic cycle, in natural and human-made hydraulic systems, and in water resources systems that include both hydrologic and hydraulic systems. The commonality of all hydrosystems is the physical laws that define the flow of fluid in these systems. A consistent mechanism for developing these physical laws is called the *control volume approach.*

The simplified concept of a system is very important in the control volume approach because of the extreme complexity of hydrosystems. Typically a system defined from the fluids viewpoint is defined as a given quantity of mass. A *system* is also a set of connected parts that form a whole. For the present discussion the fluids viewpoint will be used, in which the system has a *system boundary* or *control surface* (CS) as shown in Figure 3.2.1. A control surface is the surface that surrounds the control volume. The control surface can coincide with physical boundaries such as the wall of a pipe or the boundary of a watershed. Part of the control surface may be a hypothetical surface through which fluid flows.

Two properties, *extensive properties* and *intensive properties*, are used in the control volume approach to apply physical properties for discrete masses to a fluid flowing continuously through a control volume. Extensive properties are related to the total mass of the system (control volume), whereas intensive properties are independent of the amount of fluid. The extensive properties are mass m, momentum mV, and energy E. Corresponding intensive properties are mass per unit mass, momentum per unit mass, which is velocity v, and energy per unit mass e. In other words, for an extensive property B, the corresponding intensive property β is defined as the quantity of B per unit mass, $\beta = dB/dm$. Both the extensive and intensive properties can be scalar or vector quantities.

The relationship between intensive and extensive properties for a given system is defined by the following integral over the system:

$$B = \int_{\text{system}} \beta\, dm = \int \beta \rho\, d\forall \tag{3.2.1}$$

where dm and $d\forall$ are the differential mass and differential volume, respectively, and ρ is the fluid density.

The volume rate of flow past a given area A is expressed as

$$Q = \mathbf{V} \cdot \mathbf{A} \tag{3.2.2}$$

where $\mathbf{V}$ is the velocity, directed normal to the area and points outward from the control volume, and $\mathbf{A}$ is the area vector.

For the control volume in Figure 3.2.1 the net flowrate $\dot{Q}$ is

$$\begin{aligned} \dot{Q} &= Q_{\text{out}} - Q_{\text{in}} \\ &= \mathbf{V}_2 \cdot \mathbf{A}_2 - \mathbf{V}_1 \cdot \mathbf{A}_1 \\ &= \sum_{CS} \mathbf{V} \cdot \mathbf{A} \end{aligned} \tag{3.2.3}$$

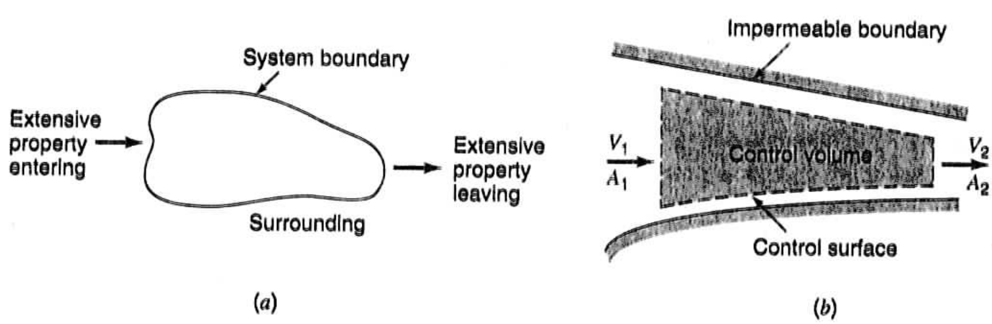

Figure 3.2.1 Control volume approach. (*a*) System and surrounding; (*b*) Control volume as a system.

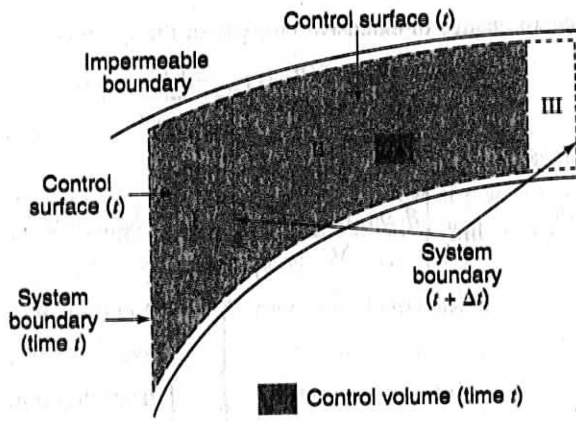

Figure 3.2.2 Control volume at times t and $t+\Delta t$.

In other words, the dot product $\mathbf{V}\cdot\mathbf{A}$ for all flows in and out of a control volume is the net rate of outflow.

The mass rate of flow out of the control volume is

$$\frac{dm}{dt}=\dot{m}=\sum_{CS}\rho\,\mathbf{V}\cdot\mathbf{A} \tag{3.2.4}$$

The rate of flow of extensive property B is the product of the mass rate and the intensive property:

$$\frac{dB}{dt}=\dot{B}=\sum_{CS}\beta\rho\,\mathbf{V}\cdot\mathbf{A} \tag{3.2.5}$$

If the velocity varies across the flow section, then it must be integrated across the section, so that the above equation for the rate of flow of extensive property $\dot{B}$ from the control volume becomes

$$\dot{B}=\int_{CS}\beta\rho\,\mathbf{V}\cdot d\mathbf{A} \tag{3.2.6}$$

Considering the system in Figure 3.2.2, the control volume is defined by the control surface at time t(I + II) with extensive property B_t. At time $t+\Delta t$ the control volume, defined by the control surface, (II + III) has moved and has extensive property $B_{t+\Delta t}$. The rate of change of extensive property B is

$$\frac{dB}{dt}=\lim_{\Delta t\to 0}\left[\frac{B_{t+\Delta t}-B_t}{\Delta t}\right] \tag{3.2.7}$$

The mass of the system at time $t+\Delta t$, $m_{sys,t+\Delta t}$, is

$$m_{sys,t+\Delta t}=m_{t+\Delta t}+\Delta m_{out}-\Delta m_{in} \tag{3.2.8}$$

where $m_{t+\Delta t}$ = mass of fluid within the control volume at time $t+\Delta t$
Δm_{out} = mass of fluid that has moved out of the control volume in time Δt
Δm_{in} = mass of fluid that has moved into the control volume in time Δt

The extensive property of the system at time $t+\Delta t$ is

$$B_{sys}=B_{CV,t+\Delta t}+\Delta B_{out}-\Delta B_{in} \tag{3.2.9}$$

where $B_{CV,t+\Delta t}$ = amount of extensive property in the control volume at time $t+\Delta t$
ΔB_{out} = amount of extensive property of the system that has moved out of the control volume in time Δt
ΔB_{in} = amount of extensive property of the system that has moved into the control volume in time Δt

The time rate of change of extensive property of the system is

$$\frac{dB_{sys}}{dt} = \lim_{\Delta t \to 0}\left[\frac{(B_{CV,t+\Delta t} + \Delta B_{out} - \Delta B_{in}) - B_{CV,t}}{\Delta t}\right] \tag{3.2.10}$$

The expression can be rearranged to yield

$$\begin{aligned}\frac{dB_{sys}}{dt} &= \lim_{\Delta t \to 0}\left[\frac{B_{CV,t+\Delta t} - B_{CV,t}}{\Delta t}\right] + \lim_{\Delta t \to 0}\left[\frac{\Delta B_{out} - \Delta B_{in}}{\Delta t}\right] \\ &= \left\{\begin{matrix}\text{Rate of change with} \\ \text{respect to time of} \\ \text{extensive property} \\ \text{in the control volume}\end{matrix}\right\} + \left\{\begin{matrix}\text{Net flow of} \\ \text{extensive property} \\ \text{from the control} \\ \text{volume}\end{matrix}\right\} \\ &= \frac{dB_{CV}}{dt} + \frac{dB}{dt}\end{aligned} \tag{3.2.11}$$

The derivative $\frac{dB_{CV}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho\, d\forall$ and $\frac{dB}{dt}$ is defined by equation (3.2.5), so that the *control volume equation for one-dimensional flow* becomes

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho\, d\forall + \sum_{CS} \beta\rho\, \mathbf{V}\cdot\mathbf{A} \tag{3.2.12}$$

The above equation for the general control volume equation was derived for one-dimensional flow so that the rate of flow of B at each section is $\beta\rho\mathbf{V}\cdot\mathbf{A}$. A more general form for rate of flow of an extensive property considers the velocity as variable across a section. Using equation (3.2.6), then, the *general control volume equation* is expressed as

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\int_{CV} \beta\rho\, d\forall + \int_{CS} \beta\rho\, \mathbf{V}\cdot d\mathbf{A} \tag{3.2.13}$$

This general control volume equation (also referred to as the *Reynolds transport theorem*) states that the total rate of change of extensive property of a flow is equal to the rate of change of extensive property stored in the control volume, $\frac{d}{dt}\int_{CV} \beta\rho\, d\forall$, plus the net rate of outflow of extensive property through the control surface, $\int_{CS} \beta\rho\, \mathbf{V}\cdot d\mathbf{A}$.

Throughout this book the general control volume equation (approach) is applied to develop continuity, energy, and momentum equations for hydrosystem (hydrologic and hydraulic) processes.

3.3 CONTINUITY

In order to write the continuity equation, the extensive property is mass ($B = m$) and the intensive property $\beta = dB/dm = 1$. By the law of conservation of mass, the mass of a system is constant, therefore $dB/dt = dm/dt = 0$. The general form of the continuity equation is then

$$0 = \frac{d}{dt}\int_{CV} \rho\, d\forall + \int_{CS} \rho\, \mathbf{V}\cdot d\mathbf{A} \tag{3.3.1}$$

which is the ***integral equation of continuity for an unsteady, variable-density flow***. Equation (3.3.1) can be rewritten as

$$\int_{CS} \rho \mathbf{V} \cdot d\mathbf{A} = \frac{d}{dt} \int_{CV} \rho \, d\forall \tag{3.3.2}$$

which states that the net rate of outflow of mass from the control volume is equal to the rate of decrease of mass within the control volume.

For flow with constant density, equation (3.3.2) can be expressed as

$$\int_{CS} \mathbf{V} \cdot d\mathbf{A} = \frac{d}{dt} \int_{CV} d\forall \tag{3.3.3}$$

The continuity equation for flow with a uniform velocity across the flow section and constant density is expressed as

$$\sum_{CS} \mathbf{V} \cdot \mathbf{A} = \frac{d}{dt} \int_{CV} d\forall \tag{3.3.4}$$

For a ***constant-density, steady one-dimensional flow***, such as water flowing in a conduit, the velocity is the mean velocity, then

$$\sum_{CS} \mathbf{V} \cdot \mathbf{A} = 0 \tag{3.3.5}$$

For pipe conduit flow we consider a control volume between two locations of the pipe, at sections 1 and 2, then the continuity equation is

$$-V_1A_1 + V_2A_2 = 0 \tag{3.3.6a}$$

or

$$V_1A_1 = V_2A_2 \tag{3.3.6b}$$

or

$$Q_1 = Q_2 \tag{3.3.6c}$$

For a ***constant-density unsteady flow***, consider the integral $\int_{CV} d\forall$ as the volume of fluid stored in a control volume denoted by S, so that

$$\frac{d}{dt} \int_{CV} d\forall = \frac{dS}{dt} \tag{3.3.7}$$

The net outflow is defined as

$$\int_{CS} \mathbf{V} \cdot d\mathbf{A} = \int_{\text{outlet}} \mathbf{V} \cdot d\mathbf{A} + \int_{\text{inlet}} \mathbf{V} \cdot d\mathbf{A} = Q(t) - I(t) \tag{3.3.8}$$

Then the integral equation of continuity is determined by substituting equations (3.3.7) and (3.3.8) into equation (3.3.2) to obtain

$$Q(t) - I(t) = -\frac{dS}{dt} \tag{3.3.9}$$

which is more commonly expressed as

$$\frac{dS}{dt} = I(t) - Q(t) \tag{3.3.10}$$

This continuity expression is used extensively in describing hydrologic processes.

EXAMPLE 3.3.1

A river section is defined by two bridges. At a particular time the flow at the upstream bridge is 100 m^3/s, and at the same time the flow at the downstream bridge is 75 m^3/s. At this particular time, what is the rate at which water is being stored in the river section, assuming no losses?

SOLUTION

Using the continuity equation (3.3.10) yields

$$\frac{dS}{dt} = Q_{up}(t) - Q_{down}(t)$$
$$= 100\ m^3/s - 75\ m^3/s$$
$$= 25\ m^3/s$$

EXAMPLE 3.3.2

A reservoir has the following monthly inflows and outflows in relative units:

Month	*J*	*F*	*M*	*A*
Inflows	10	5	0	5
Outflows	5	5	10	0

If the reservoir contains 30 units of water in storage at the beginning of the year, how many units of water in storage are there at the end of April?

SOLUTION

The continuity equation (3.3.10) is used to perform a routing of flows into and out of the reservoir. Because the inflow and outflows are for discrete time intervals, the continuity equation (3.3.10) can be reformulated as

$$dS = I(t)dt - Q(t)dt$$

and integrated over time intervals $j = 1, 2, \ldots J$ of each length Δt:

$$\int_{S_{j-1}}^{S_j} dS = \int_{(j-1)\Delta t}^{j\Delta t} I(t)dt - \int_{(j-1)\Delta t}^{j\Delta t} Q(t)dt$$

or

$$S_j - S_{j-1} = I_j - Q_j \text{ for } j = 1, 2, \ldots$$
$$\Delta S_j = I_j - Q_j$$

where I_j and Q_j are the volumes of inflow and outflow for the jth time interval. The cumulative storage is $S_{j+1} = S_j + \Delta S_j$. For the first interval of time,

$$\Delta S_1 = I_1 - Q_1 = 10 - 5 = 5$$

Then $S_2 = S_1 + \Delta S_1 = 30 + 5 = 35$. The remaining computations are:

Time	I_j	Q_j	ΔS_j	S_j
1	10	5	5	30
2	5	5	0	35
3	0	10	−10	25
4	5	0	5	30

3.4 ENERGY

This section uses the first law of thermodynamics along with the control volume approach to develop the energy equation for fluid flow in hydrologic and hydraulic processes. An energy balance for

VP and Publisher Don Fowley
Acquisition Editor Jenny Welter
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Production Manager Janis Soo
Assistant Production Editor Elaine S. Chew
Senior Marketing Manager Christopher Ruel
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Media Editor Lauren Sapira
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Cover Image © Larry W. Mays

This book was set in 9.5/12 Times Roman by Thomson Digital and printed and bound by Hamilton Printing Company. The cover was printed by Hamilton Printing Company.

This book is printed on acid free paper.

Library of Congress Cataloging in Publication Data

Mays, Larry W.
Water resources engineering / Larry W. Mays.—2nd ed.
p. cm.
Includes index.
ISBN 978-0-470-46064-1 (cloth : alk. paper)
1. Hydraulic engineering. 2. Hydrology. I. Title.

TC145.M383 2010
627—dc22 2010005952

Printed in the United States of America
10 9 8 7 6 5