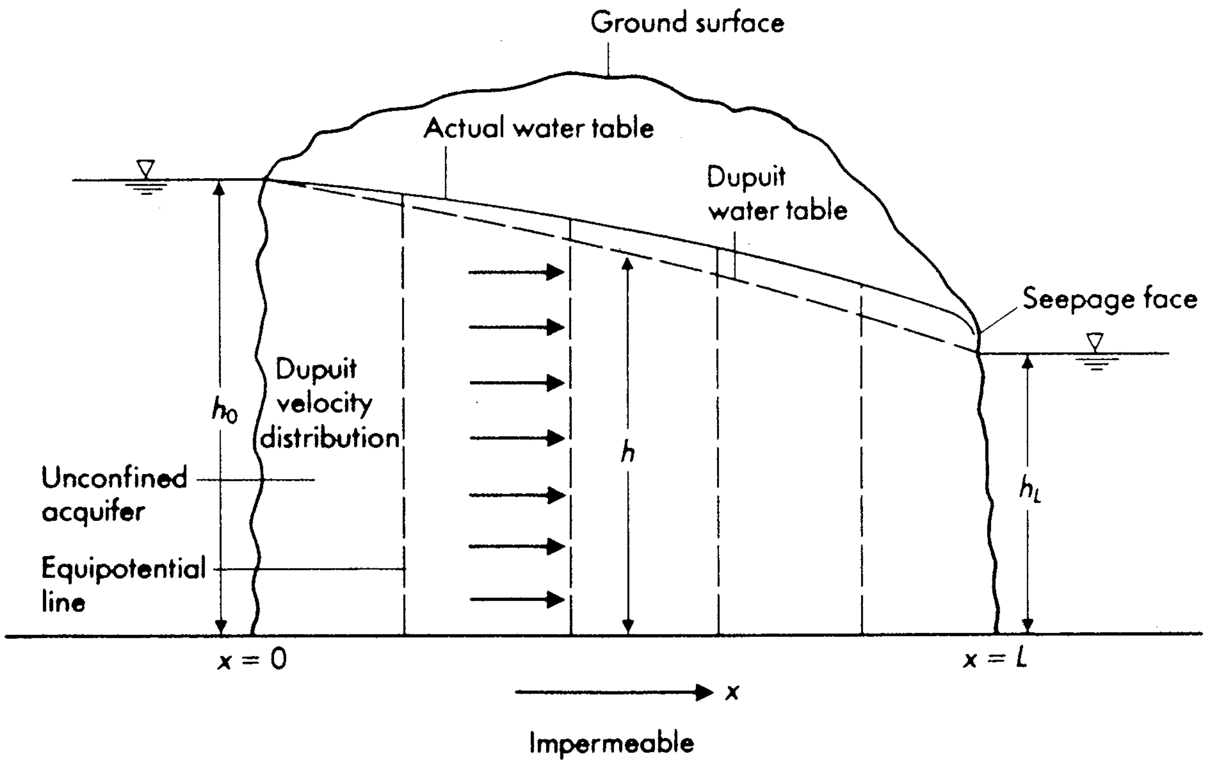

Figure 2.16 Steady flow in an unconfined aquifer between two water bodies with vertical boundaries. Source: Bedient and Huber, 1992.

SOURCE: Bedient et al., 1999

Example 2.2 DUPUIT EQUATION

Derive the equation for one-dimensional flow in an unconfined aquifer using the Dupuit assumptions (Figure 2.16).

Solution. Darcy's law gives the one-dimensional flow per unit width as

$$q = -Kh\frac{dh}{dx}$$

where h and x are as defined in Figure 2.16. At steady state, the rate of change of q with distance is zero, or

$$\frac{d}{dx}\left(-Kh\frac{dh}{dx}\right) = 0$$

$$-\frac{K}{2}\frac{d^2h^2}{dx^2} = 0$$

or

$$\frac{d^2h^2}{dx^2} = 0$$

Integration yields

$$h^2 = ax + b$$

where a and b are constants. Setting the boundary condition $h = h_0$ at $x = 0$,

$$b = h_0^2$$

Differentiation of $h^2 = ax + b$ gives

$$a = 2h\frac{dh}{dx}$$

From Darcy's law,

$$h\frac{dh}{dx} = -\frac{q}{K}$$

so, by substitution,

$$h^2 = h_0^2 - \frac{2qx}{K}$$

Setting $h = h_L$ at $x = L$ and neglecting flow across the seepage face yields

$$h_L^2 = h_0^2 - \frac{2qL}{K}$$

Rearrangement gives

$$\boxed{q = \frac{K}{2L}(h_0^2 - h_L^2), \qquad \text{Dupuit equation}}$$

SOURCE: Bedient et al., 1994

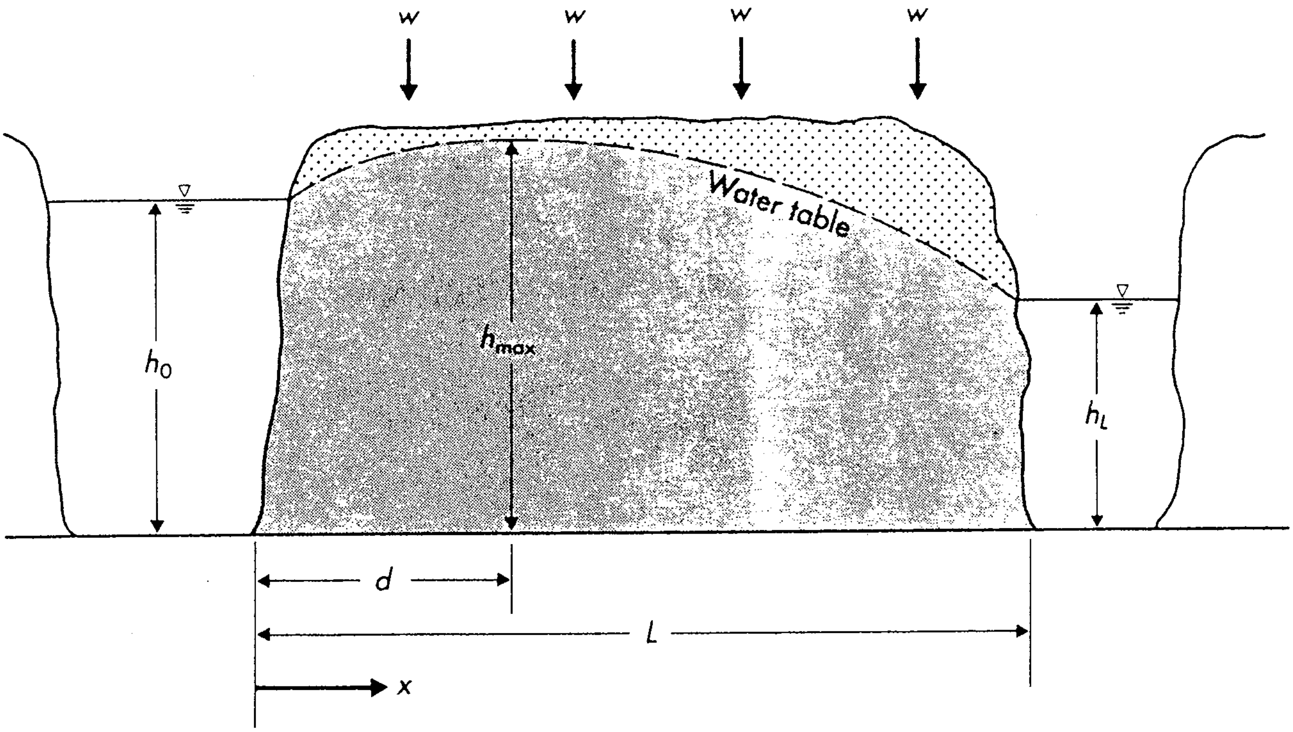

Figure 2.17 Dupuit parabola with recharge. Source: Bedient and Huber, 1992.

SOURCE : Bedient et al. , 1999

Example 2.3 DUPUIT EQUATION WITH RECHARGE

(a) *Derive the general Dupuit equation with the effect of recharge.*

(b) Two rivers located 1000 m apart fully penetrate an aquifer (Figure 2.17). The aquifer has a K value of 0.5 m/day. The region receives an average rainfall of 15 cm/yr and evaporation is about 10 cm/yr. Assume that the water elevation in river 1 is 20 m and the water elevation in river 2 is 18 m. Using the equation derived in part (a), determine the location and height of the water divide.

(c) What is the daily discharge per meter of width into each river?

Solution

(a) Designating recharge intensity as W, it can be seen that

$$\frac{dq}{dx} = W$$

Source: Bedient et al., 1999

From Darcy's law for one-dimensional flow, the flow per unit width is

$$q = -Kh\frac{dh}{dx}$$

Substituting the second equation into the first yields

$$-\frac{K}{2}\frac{d^2h^2}{dx^2} = W$$

or

$$\frac{d^2h^2}{dx^2} = -\frac{2W}{K}$$

Integration gives

$$h^2 = -\frac{Wx^2}{K} + ax + b$$

where a and b are constants. The boundary condition $h = h_0$ at $x = 0$ gives

$$b = h_0^2$$

and the boundary condition $h = h_L$ at $x = L$ gives

$$a = \frac{h_L^2 - h_0^2}{L} + \frac{WL}{K}$$

Substitution of a and b into the previous equation for h^2 yields

$$\boxed{h^2 = h_0^2 + \frac{h_L^2 - h_0^2}{L}x + \frac{Wx}{K}(L - x), \qquad \text{Dupuit parabola}}$$

This equation will give the shape of the Dupuit parabola shown in Figure 2.17. If $W = 0$, this equation will reduce to the parabolic equation found in Example 2.2. Differentiation of the parabolic equation gives

$$2h\frac{dh}{dx} = \frac{h_L^2 - h_0^2}{L} + \frac{W}{K}(L - 2x)$$

But Darcy's law gives

$$h\frac{dh}{dx} = -\frac{q}{K}$$

so

$$-2\frac{q}{K} = \frac{1}{L}(h_L^2 - h_0^2) + \frac{W}{K}(L - 2x)$$

Simplifying,

$$\boxed{q = \frac{K}{2L}(h_0^2 - h_L^2) + W\left(x - \frac{L}{2}\right)}$$

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(b) Given

$$L = 100 \text{ m}$$
$$K = 0.5 \text{ m/day}$$
$$h_0 = 20 \text{ m}$$
$$h_L = 18 \text{ m}$$
$$W = 5 \text{ cm/yr} = 1.369 \times 10^{-4} \text{ m/day}$$

At $x = d$, $q = 0$ (see Figure 2.17),

$$0 = \frac{K}{2L}(h_0^2 - h_L^2) + W\left(d - \frac{L}{2}\right)$$

$$d = \frac{L}{2} - \frac{K}{2WL}(h_0^2 - h_L^2)$$
$$= \frac{1000 \text{ m}}{2} - \frac{(0.5 \text{ m/day})(20^2 \text{ m}^2 - 18^2 \text{ m}^2)}{(2)(1.369 \times 10^{-4} \text{ m/day})(1000 \text{ m})}$$
$$= 500 \text{ m} - 138.8 \text{ m}$$

$$\boxed{d = 361.2 \text{ m}}$$

At $x = d$, $h = h_{max}$,

$$h_{max}^2 = h_0^2 + \frac{h_L^2 - h_0^2}{L}d + \frac{Wd}{K}(L - d)$$
$$= (20 \text{ m})^2 + \frac{(18^2 \text{ m}^2 - 20^2 \text{ m}^2)}{1000 \text{ m}} 361.2 \text{ m}$$
$$+ \frac{(1.369 \times 10^{-4} \text{ m/day})(361.2 \text{ m})}{0.5 \text{ m/day}}(1000 \text{ m} - 361.2 \text{ m})$$
$$= 400 \text{ m}^2 - 27.5 \text{ m}^2 + 63.2 \text{ m}^2$$
$$= 435.7 \text{ m}^2$$

$$\boxed{h_{max} = 20.9 \text{ m}}$$

Thus, the water divide is located 361.2 m from the edge of river 1 and is 20.9 m high.

(c) For discharge into river 1, set $x = 0$ m:

$$q = \frac{K}{2L}(h_0^2 - h_L^2) + W\left(0 - \frac{L}{2}\right)$$
$$= \frac{0.5 \text{ m/day}}{(2)(1000 \text{ m})}(20^2 \text{ m}^2 - 18^2 \text{ m}^2)$$
$$+ (1.369 \times 10^{-4} \text{ m/day})(-1000 \text{ m/2})$$
$$= -0.0495 \text{ m}^2\text{/day}$$

The negative sign indicates that flow is in the opposite direction from the x direction. Therefore,

$$\boxed{q = 0.0495 \text{ m}^2\text{/day into river 1}}$$

For discharge into river 2, set $x = L = 1000$ m:

$$q = \frac{K}{2L}(h_0^2 - h_L^2) + W\left(1000 \text{ m} - \frac{L}{2}\right)$$
$$= \frac{0.5 \text{ m/day}}{(2)(1000 \text{ m})}(20^2 \text{ m}^2 - 18^2 \text{ m}^2)$$
$$+ (1.369 \times 10^{-4} \text{ m/day})(1000 \text{ m} - 1000 \text{ m/2})$$

$$\boxed{q = 0.08745 \text{ m}^2\text{/day into river 2}}$$

Source: Bedient et al., 1999

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