Homework 5
Due June 17

6.1
1, 2, 10, 14, 15, 19

6.2
1, 5, 7, 9

6.3
1, 3, 7

which can be rearranged to produce

$$\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\vartheta = \frac{1}{2}\left[\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2\right]$$

$$= \frac{1}{2}\left[u_1^2 + u_2^2 + v_1^2 + v_2^2 - (u_1-v_1)^2 - (u_2-v_2)^2\right]$$

$$= u_1 v_1 + u_2 v_2$$

$$= \mathbf{u}\cdot\mathbf{v}$$

The verification for $\mathbb{R}^3$ is similar. When $n > 3$, formula (2) may be used to *define the* angle between two vectors in $\mathbb{R}^n$. In statistics, for instance, the value of $\cos\vartheta$ defined by (2) for suitable vectors $\mathbf{u}$ and $\mathbf{v}$ is what statisticians call a *correlation coefficient*.

PRACTICE PROBLEMS

Let $\mathbf{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$.

1. Compute $\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}$ and $\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$.

2. Find a unit vector $\mathbf{u}$ in the direction of $\mathbf{c}$.

3. Show that $\mathbf{d}$ is orthogonal to $\mathbf{c}$.

4. Use the results of Practice Problems 2 and 3 to explain why $\mathbf{d}$ must be orthogonal to the unit vector $\mathbf{u}$.

6.1 EXERCISES

Compute the quantities in Exercises 1–8 using the vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 4 \\ 6 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$$

1. $\mathbf{u}\cdot\mathbf{u}$, $\mathbf{v}\cdot\mathbf{u}$, and $\dfrac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}$

2. $\mathbf{w}\cdot\mathbf{w}$, $\mathbf{x}\cdot\mathbf{w}$, and $\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}$

3. $\dfrac{1}{\mathbf{w}\cdot\mathbf{w}}\mathbf{w}$

4. $\dfrac{1}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$

5. $\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$

6. $\left(\dfrac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}$

7. $\|\mathbf{w}\|$

8. $\|\mathbf{x}\|$

In Exercises 9–12, find a unit vector in the direction of the given vector.

9. $\begin{bmatrix} -30 \\ 40 \end{bmatrix}$

10. $\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$

11. $\begin{bmatrix} 7/4 \\ 1/2 \\ 1 \end{bmatrix}$

12. $\begin{bmatrix} 8/3 \\ 2 \end{bmatrix}$

13. Find the distance between $\mathbf{x} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$.

14. Find the distance between $\mathbf{u} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -4 \\ -1 \\ 8 \end{bmatrix}$.

Determine which pairs of vectors in Exercises 15–18 are orthogonal.

15. $\mathbf{a} = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

16. $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

17. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -4 \\ 1 \\ -2 \\ 6 \end{bmatrix}$

18. $\mathbf{y} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

In Exercises 19 and 20, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

19. a. $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$.

   b. For any scalar c, $\mathbf{u}\cdot(c\mathbf{v}) = c(\mathbf{u}\cdot\mathbf{v})$.

   c. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance from $\mathbf{u}$ to $-\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

   d. For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.

   e. If vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ span a subspace W and if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ for $j = 1, \ldots, p$, then $\mathbf{x}$ is in $W^\perp$.

20. a. $\mathbf{u}\cdot\mathbf{v} - $

   b. For any

   c. If $\mathbf{x}$ is o: is in W

   d. If $\|\mathbf{u}\|^2$

   e. For an n orthogo:

21. Use the tra parts (b) an from Chapt

22. Let $\mathbf{u} = (u$ $\mathbf{u}\cdot\mathbf{u} = 0$?

23. Let $\mathbf{u} = $ $\mathbf{u}\cdot\mathbf{v}$, $\|\mathbf{u}\|^2$, Theorem.

24. Verify the p $\|\mathbf{u}+\mathbf{v}\|^2 +$

25. Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ orthogonal t

26. Let $\mathbf{u} = $ $\mathbf{u}\cdot\mathbf{x} = 0$. WI W is a subsp

27. Suppose a ve that $\mathbf{y}$ is ortho

28. Suppose $\mathbf{y}$ is thogonal to e in Span $\{\mathbf{u}, \mathbf{v}$ orthogonal to

29. Let $W = $ Spa each $\mathbf{v}_j$, for 1 in W.

20. a. $\mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} = 0$.

b. For any scalar c, $\|c\mathbf{v}\| = c\|\mathbf{v}\|$.

c. If $\mathbf{x}$ is orthogonal to every vector in a subspace W, then $\mathbf{x}$ is in $W^{\perp}$.

d. If $\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

e. For an $m \times n$ matrix A, vectors in the null space of A are orthogonal to vectors in the row space of A.

21. Use the transpose definition of the inner product to verify parts (b) and (c) of Theorem 1. Mention the appropriate facts from Chapter 2.

22. Let $\mathbf{u} = (u_1, u_2, u_3)$. Explain why $\mathbf{u} \cdot \mathbf{u} \geq 0$. When is $\mathbf{u} \cdot \mathbf{u} = 0$?

23. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -7 \\ -4 \\ 6 \end{bmatrix}$. Compute and compare $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|^2$, $\|\mathbf{v}\|^2$, and $\|\mathbf{u} + \mathbf{v}\|^2$. Do not use the Pythagorean Theorem.

24. Verify the *parallelogram law* for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:
$$\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$$

25. Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. Describe the set H of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ that are orthogonal to $\mathbf{v}$. [*Hint:* Consider $\mathbf{v} = \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$.]

26. Let $\mathbf{u} = \begin{bmatrix} 5 \\ -6 \\ 7 \end{bmatrix}$, and let W be the set of all $\mathbf{x}$ in $\mathbb{R}^3$ such that $\mathbf{u} \cdot \mathbf{x} = 0$. What theorem in Chapter 4 can be used to show that W is a subspace of $\mathbb{R}^3$? Describe W in geometric language.

27. Suppose a vector $\mathbf{y}$ is orthogonal to vectors $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to the vector $\mathbf{u} + \mathbf{v}$.

28. Suppose $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$. [*Hint:* An arbitrary $\mathbf{w}$ in Span $\{\mathbf{u}, \mathbf{v}\}$ has the form $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to such a vector $\mathbf{w}$.]
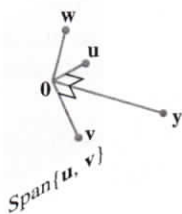
29. Let $W = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Show that if $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$, for $1 \leq j \leq p$, then $\mathbf{x}$ is orthogonal to every vector in W.

30. Let W be a subspace of $\mathbb{R}^n$, and let $W^{\perp}$ be the set of all vectors orthogonal to W. Show that $W^{\perp}$ is a subspace of $\mathbb{R}^n$ using the following steps.

a. Take $\mathbf{z}$ in $W^{\perp}$, and let $\mathbf{u}$ represent any element of W. Then $\mathbf{z} \cdot \mathbf{u} = 0$. Take any scalar c and show that $c\mathbf{z}$ is orthogonal to $\mathbf{u}$. (Since $\mathbf{u}$ was an arbitrary element of W, this will show that $c\mathbf{z}$ is in $W^{\perp}$.)

b. Take $\mathbf{z}_1$ and $\mathbf{z}_2$ in $W^{\perp}$, and let $\mathbf{u}$ be any element of W. Show that $\mathbf{z}_1 + \mathbf{z}_2$ is orthogonal to $\mathbf{u}$. What can you conclude about $\mathbf{z}_1 + \mathbf{z}_2$? Why?

c. Finish the proof that $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

31. Show that if $\mathbf{x}$ is in both W and $W^{\perp}$, then $\mathbf{x} = \mathbf{0}$.

32. [M] Construct a pair $\mathbf{u}$, $\mathbf{v}$ of random vectors in $\mathbb{R}^4$, and let

$$A = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .5 & .5 & -.5 & -.5 \\ .5 & -.5 & .5 & -.5 \\ .5 & -.5 & -.5 & .5 \end{bmatrix}$$

a. Denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_4$. Compute the length of each column, and compute $\mathbf{a}_1 \cdot \mathbf{a}_2$, $\mathbf{a}_1 \cdot \mathbf{a}_3$, $\mathbf{a}_1 \cdot \mathbf{a}_4$, $\mathbf{a}_2 \cdot \mathbf{a}_3$, $\mathbf{a}_2 \cdot \mathbf{a}_4$, and $\mathbf{a}_3 \cdot \mathbf{a}_4$.

b. Compute and compare the lengths of $\mathbf{u}$, $A\mathbf{u}$, $\mathbf{v}$, and $A\mathbf{v}$.

c. Use equation (2) in this section to compute the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$. Compare this with the cosine of the angle between $A\mathbf{u}$ and $A\mathbf{v}$.

d. Repeat parts (b) and (c) for two other pairs of random vectors. What do you conjecture about the effect of A on vectors?

33. [M] Generate random vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{v}$ in $\mathbb{R}^4$ with integer entries (and $\mathbf{v} \neq \mathbf{0}$), and compute the quantities

$$\left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}, \quad \left(\frac{\mathbf{y} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}, \quad \frac{(\mathbf{x} + \mathbf{y}) \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}, \quad \frac{(10\mathbf{x}) \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$$

Repeat the computations with new random vectors $\mathbf{x}$ and $\mathbf{y}$. What do you conjecture about the mapping $\mathbf{x} \mapsto T(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\mathbf{v}$ (for $\mathbf{v} \neq \mathbf{0}$)? Verify your conjecture algebraically.

34. [M] Let $A = \begin{bmatrix} -6 & 3 & -27 & -33 & -13 \\ 6 & -5 & 25 & 28 & 14 \\ 8 & -6 & 34 & 38 & 18 \\ 12 & -10 & 50 & 41 & 23 \\ 14 & -21 & 49 & 29 & 33 \end{bmatrix}$. Construct a matrix N whose columns form a basis for Nul A, and construct a matrix R whose *rows* form a basis for Row A (see Section 4.6 for details). Perform a matrix computation with N and R that illustrates a fact from Theorem 3.

SOLUTION

$$U\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\|U\mathbf{x}\| = \sqrt{9+1+1} = \sqrt{11}$$

$$\|\mathbf{x}\| = \sqrt{2+9} = \sqrt{11}$$

Theorems 6 and 7 are particularly useful when applied to *square* matrices. An **orthogonal matrix** is a square invertible matrix U such that $U^{-1} = U^T$. By Theorem 6, such a matrix has orthonormal columns.[1] It is easy to see that any *square* matrix with orthonormal columns is an orthogonal matrix. Surprisingly, such a matrix must have orthonormal *rows*, too. See Exercises 27 and 28. Orthogonal matrices will appear frequently in Chapter 7.

EXAMPLE 7 The matrix

$$U = \begin{bmatrix} 3/\sqrt{11} & -1/\sqrt{6} & -1/\sqrt{66} \\ 1/\sqrt{11} & 2/\sqrt{6} & -4/\sqrt{66} \\ 1/\sqrt{11} & 1/\sqrt{6} & 7/\sqrt{66} \end{bmatrix}$$

is an orthogonal matrix because it is square and because its columns are orthonormal, by Example 5. Verify that the rows are orthonormal, too! ∎

PRACTICE PROBLEMS

1. Let $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.

2. Let $\mathbf{y}$ and L be as in Example 3 and Fig. 3. Compute the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto L using $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ instead of the $\mathbf{u}$ in Example 3.

3. Let U and $\mathbf{x}$ be as in Example 6, and let $\mathbf{y} = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$. Verify that $U\mathbf{x} \cdot U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.

6.2 EXERCISES

In Exercises 1–6, determine which sets of vectors are orthogonal.

1. $\begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix}$
2. $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix}$

5. $\begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix}$
6. $\begin{bmatrix} 5 \\ -4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 5 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 2 \\ -7 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ -3 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$
4. $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}$

In Exercises 7–10, show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ or $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal basis for $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively. Then express $\mathbf{x}$ as a linear combination of the $\mathbf{u}$'s.

7. $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 9 \\ -7 \end{bmatrix}$

[1] A better name might be *orthonormal matrix*, and this term is found in some statistics texts. However, *orthogonal matrix* is the standard term in linear algebra.

8. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

9. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

10. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$

11. Compute t through $\begin{bmatrix} - \end{bmatrix}$

12. Compute th through $\begin{bmatrix} - \end{bmatrix}$

13. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ orthogonal

14. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ in Span $\{\mathbf{u}\}$

15. Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ to the line th

16. Let $\mathbf{y} = \begin{bmatrix} - \\ \end{bmatrix}$ to the line thi

In Exercises 17–2 mal. If a set is onl an orthonormal se

17. $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} - \end{bmatrix}$

19. $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}, \begin{bmatrix} .8 \\ .6 \end{bmatrix}$

21. $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix},$

22. $\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix},$

In Exercises 23 and True or False. Justi

23. a. Not every l set.

8. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$

9. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}$

10. $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

11. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin.

12. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ onto the line through $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$ and the origin.

13. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ -7 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of two orthogonal vectors, one in Span $\{\mathbf{u}\}$ and one orthogonal to $\mathbf{u}$.

14. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Write $\mathbf{y}$ as the sum of a vector in Span $\{\mathbf{u}\}$ and a vector orthogonal to $\mathbf{u}$.

15. Let $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

16. Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{u}$ and the origin.

In Exercises 17–22, determine which sets of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

17. $\begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$, $\begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$

18. $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

19. $\begin{bmatrix} -.6 \\ .8 \end{bmatrix}$, $\begin{bmatrix} .8 \\ .6 \end{bmatrix}$

20. $\begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\begin{bmatrix} 1/3 \\ 2/3 \\ 0 \end{bmatrix}$

21. $\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix}$, $\begin{bmatrix} 3/\sqrt{10} \\ -1/\sqrt{20} \\ -1/\sqrt{20} \end{bmatrix}$, $\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

22. $\begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$, $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$, $\begin{bmatrix} -2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

In Exercises 23 and 24, all vectors are in $\mathbb{R}^n$. Mark each statement True or False. Justify each answer.

23. a. Not every linearly independent set in $\mathbb{R}^n$ is an orthogonal set.

b. If $\mathbf{y}$ is a linear combination of nonzero vectors from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.

c. If the vectors in an orthogonal set of nonzero vectors are normalized, then some of the new vectors may not be orthogonal.

d. A matrix with orthonormal columns is an orthogonal matrix.

e. If L is a line through $\mathbf{0}$ and if $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L.

24. a. Not every orthogonal set in $\mathbb{R}^n$ is linearly independent.

b. If a set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ has the property that $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$, then S is an orthonormal set.

c. If the columns of an $m \times n$ matrix A are orthonormal, then the linear mapping $\mathbf{x} \mapsto A\mathbf{x}$ preserves lengths.

d. The orthogonal projection of $\mathbf{y}$ onto $\mathbf{v}$ is the same as the orthogonal projection of $\mathbf{y}$ onto $c\mathbf{v}$ whenever $c \neq 0$.

e. An orthogonal matrix is invertible.

25. Prove Theorem 7. [*Hint:* For (a), compute $\|U\mathbf{x}\|^2$, or prove (b) first.]

26. Suppose W is a subspace of $\mathbb{R}^n$ spanned by n nonzero orthogonal vectors. Explain why $W = \mathbb{R}^n$.

27. Let U be a square matrix with orthonormal columns. Explain why U is invertible. (Mention the theorems you use.)

28. Let U be an $n \times n$ orthogonal matrix. Show that the rows of U form an orthonormal basis of $\mathbb{R}^n$.

29. Let U and V be $n \times n$ orthogonal matrices. Explain why UV is an orthogonal matrix. [That is, explain why UV is invertible and its inverse is $(UV)^T$.]

30. Let U be an orthogonal matrix, and construct V by interchanging some of the columns of U. Explain why V is an orthogonal matrix.

31. Show that the orthogonal projection of a vector $\mathbf{y}$ onto a line L through the origin in $\mathbb{R}^2$ does not depend on the choice of the nonzero $\mathbf{u}$ in L used in the formula for $\hat{\mathbf{y}}$. To do this, suppose $\mathbf{y}$ and $\mathbf{u}$ are given and $\hat{\mathbf{y}}$ has been computed by formula (2) in this section. Replace $\mathbf{u}$ in that formula by $c\mathbf{u}$, where c is an unspecified nonzero scalar. Show that the new formula gives the same $\hat{\mathbf{y}}$.

32. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthogonal set of nonzero vectors, and let c_1, c_2 be any nonzero scalars. Show that $\{c_1\mathbf{v}_1, c_2\mathbf{v}_2\}$ is also an orthogonal set. Since orthogonality of a set is defined in terms of pairs of vectors, this shows that if the vectors in an orthogonal set are normalized, the new set will still be orthogonal.

33. Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = $ Span $\{\mathbf{u}\}$. Show that the mapping $\mathbf{x} \mapsto \text{proj}_L \mathbf{x}$ is a linear transformation.

34. Given $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^n$, let $L = $ Span $\{\mathbf{u}\}$. For $\mathbf{y}$ in $\mathbb{R}^n$, the **reflection of $\mathbf{y}$ in L** is the point $\text{refl}_L \, \mathbf{y}$ defined by

PRACTICE PROBLEM

Let $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Use the fa

that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to compute $\text{proj}_W \mathbf{y}$.

6.3 EXERCISES

In Exercises 1 and 2, you may assume that $\{\mathbf{u}_1, \ldots, \mathbf{u}_4\}$ is an orthogonal basis for $\mathbb{R}^4$.

1. $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -4 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} 5 \\ -3 \\ -1 \\ 1 \end{bmatrix}$,

$\mathbf{x} = \begin{bmatrix} 10 \\ -8 \\ 2 \\ 0 \end{bmatrix}$. Write $\mathbf{x}$ as the sum of two vectors, one in

Span $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and the other in Span $\{\mathbf{u}_4\}$.

2. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}$, $\mathbf{u}_4 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$,

$\mathbf{v} = \begin{bmatrix} 4 \\ 5 \\ -3 \\ 3 \end{bmatrix}$. Write $\mathbf{v}$ as the sum of two vectors, one in

Span $\{\mathbf{u}_1\}$ and the other in Span $\{\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$.

In Exercises 3–6, verify that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, and then find the orthogonal projection of $\mathbf{y}$ onto Span $\{\mathbf{u}_1, \mathbf{u}_2\}$.

3. $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

4. $\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$

5. $\mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

6. $\mathbf{y} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 7–10, let W be the subspace spanned by the $\mathbf{u}$'s, and write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

7. $\mathbf{y} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$

8. $\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$

9. $\mathbf{y} = \begin{bmatrix} 4 \\ 3 \\ 3 \\ -1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

10. $\mathbf{y} = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

In Exercises 11 and 12, find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

11. $\mathbf{y} = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

12. $\mathbf{y} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}$

In Exercises 13 and 14, find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

13. $\mathbf{z} = \begin{bmatrix} 3 \\ -7 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

14. $\mathbf{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$

15. Let $\mathbf{y} = \begin{bmatrix} 5 \\ -9 \\ 5 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. Find the

distance from $\mathbf{y}$ to the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

16. Let $\mathbf{y}$, $\mathbf{v}_1$, and $\mathbf{v}_2$ be as in Exercise 12. Find the distance from $\mathbf{y}$ to the subspace of $\mathbb{R}^4$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

17. Let $\mathbf{y} = \begin{bmatrix} 4 \\ 8 \\ 1 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$, and

$W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

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