HW #3

4.1

9, 10, 11, 13

4.2

3, 11, 15, 25

4.3

9, 13, 19, 26

4.1

b. Find specific vector u and v in W such that $\mathbf{u}+\mathbf{v}$ is not in W. This is enough to show that W is *not* a vector space.

3. Let H be the set of points inside and on the unit circle in the xy-plane. That is, let $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 \le 1 \right\}$. Find a specific example—two vectors or a vector and a scalar—to show that H is not a subspace of $\mathbb{R}^2$.

4. Construct a geometric figure that illustrates why a line in $\mathbb{R}^2$ *not* through the origin is not closed under vector addition.

In Exercises 5–8, determine if the given set is a subspace of $\mathbb{P}_n$ for an appropriate value of n. Justify your answers.

5. All polynomials of the form $\mathbf{p}(t) = at^2$, where a is in $\mathbb{R}$.

6. All polynomials of the form $\mathbf{p}(t) = a + t^2$, where a is in $\mathbb{R}$.

7. All polynomials of degree at most 3, with integers as coefficients.

8. All polynomials in $\mathbb{P}_n$ such that $\mathbf{p}(0) = 0$.

9. Let H be the set of all vectors of the form $\begin{bmatrix} -2t \\ 5t \\ 3t \end{bmatrix}$. Find a vector $\mathbf{v}$ in $\mathbb{R}^3$ such that $H = \text{Span}\,\{\mathbf{v}\}$. Why does this show that H is a subspace of $\mathbb{R}^3$?

10. Let H be the set of all vectors of the form $\begin{bmatrix} 3t \\ 0 \\ -7t \end{bmatrix}$, where t is any real number. Show that H is a subspace of $\mathbb{R}^3$. (Use the method of Exercise 9.)

11. Let W be the set of all vectors of the form $\begin{bmatrix} 2b+3c \\ -b \\ 2c \end{bmatrix}$, where b and c are arbitrary. Find vectors u and v such that $W = \text{Span}\,\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

12. Let W be the set of all vectors of the form $\begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix}$. Show that W is a subspace of $\mathbb{R}^4$. (Use the method of Exercise 11.)

13. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

a. Is w in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? How many vectors are in $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

b. How many vectors are in Span $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

c. Is w in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

14. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be as in Exercise 13, and let $\mathbf{w} = \begin{bmatrix} 1 \\ 3 \\ 14 \end{bmatrix}$. Is w in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$? Why?

In Exercises 15–18, let W be the set of all vectors of the form shown, where a, b, and c represent arbitrary real numbers. In each case, either find a set S of vectors that spans W or give an example to show that W is *not* a vector space.

15. $\begin{bmatrix} 2a+3b \\ -1 \\ 2a-5b \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 3a-5b \\ 3b+2a \end{bmatrix}$

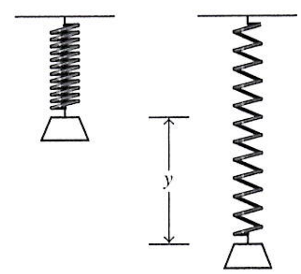
17. $\begin{bmatrix} 2a-b \\ 3b-c \\ 3c-a \\ 3b \end{bmatrix}$

18. $\begin{bmatrix} 4a+3b \\ 0 \\ a+3b+c \\ 3b-2c \end{bmatrix}$

19. If a mass m is placed at the end of a spring, and if the mass is pulled downward and released, the mass–spring system will begin to oscillate. The displacement y of the mass from its resting position is given by a function of the form

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega t \qquad (5)$$

where ω is a constant that depends on the spring and the mass. (See the figure below.) Show that the set of all functions described in (5) (with ω fixed and c_1, c_2 arbitrary) is a vector space.



20. The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$. This set is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

a. What facts about continuous functions should be proved in order to demonstrate that $C[a, b]$ is indeed a subspace as claimed? (These facts are usually discussed in a calculus class.)

b. Show that $\{\mathbf{f} \text{ in } C[a, b] : \mathbf{f}(a) = \mathbf{f}(b)\}$ is a subspace of $C[a, b]$.

For fixed positive integers m and n, the set $M_{m \times n}$ of all $m \times n$ matrices is a vector space, under the usual operations of addition of matrices and multiplication by real scalars.

21. Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times 2}$.

22. Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $M_{2\times 4}$ with the property that $FA = 0$ (the zero matrix in $M_{3\times 4}$). Determine if H is a subspace of $M_{2\times 4}$.

4.2

In Exercises 3–6, find an explicit description of Nul A, by listing vectors that span the null space.

3. $A = \begin{bmatrix} 1 & 2 & 4 & 0 \\ 0 & 1 & 3 & -2 \end{bmatrix}$

4. $A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$

6. $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 7–14, either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

7. $\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 2 \right\}$

8. $\left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} : 3r - 2 = 3s + t \right\}$

9. $\left\{ \begin{bmatrix} p \\ q \\ r \\ s \end{bmatrix} : \begin{matrix} p - 3q = 4s \\ 2p = s + 5r \end{matrix} \right\}$

10. $\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{matrix} 3a + b = c \\ a + b + 2c = 2d \end{matrix} \right\}$

11. $\left\{ \begin{bmatrix} s - 2t \\ 3 + 3s \\ 3s + t \\ 2s \end{bmatrix} : s, t \text{ real} \right\}$

12. $\left\{ \begin{bmatrix} 3p - 5q \\ 4q \\ p \\ q + 1 \end{bmatrix} : p, q \text{ real} \right\}$

13. $\left\{ \begin{bmatrix} c - 6d \\ d \\ c \end{bmatrix} : c, d \text{ real} \right\}$

14. $\left\{ \begin{bmatrix} -s + 3t \\ s - 2t \\ 5s - t \end{bmatrix} : s, t \text{ real} \right\}$

In Exercises 15 and 16, find A such that the given set is Col A.

15. $\left\{ \begin{bmatrix} 2s + t \\ r - s + 2t \\ 3r + s \\ 2r - s - t \end{bmatrix} : r, s, t \text{ real} \right\}$

16. $\left\{ \begin{bmatrix} b - c \\ 2b + 3d \\ b + 3c - 3d \\ c + d \end{bmatrix} : b, c, d \text{ real} \right\}$

For the matrices in Exercises 17–20, (a) find k such that Nul A is a subspace of $\mathbb{R}^k$, and (b) find k such that Col A is a subspace of $\mathbb{R}^k$.

17. $A = \begin{bmatrix} 6 & -4 \\ -3 & 2 \\ -9 & 6 \\ 9 & -6 \end{bmatrix}$

18. $A = \begin{bmatrix} 5 & -2 & 3 \\ -1 & 0 & -1 \\ 0 & -2 & -2 \\ -5 & 7 & 2 \end{bmatrix}$

19. $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

20. $A = \begin{bmatrix} 1 & -3 & 2 & 0 & -5 \end{bmatrix}$

21. With A as in Exercise 17, find a nonzero vector in Nul A and a nonzero vector in Col A.

22. With A as in Exercise 18, find a nonzero vector in Nul A and a nonzero vector in Col A.

23. Let $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

24. Let $A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$. Determine if $\mathbf{w}$ is in Col A. Is $\mathbf{w}$ in Nul A?

In Exercises 25 and 26, A denotes an $m \times n$ matrix. Mark each statement True or False. Justify each answer.

25. a. The null space of A is the solution set of the equation $A\mathbf{x} = \mathbf{0}$.
 b. The null space of an $m \times n$ matrix is in $\mathbb{R}^m$.
 c. The column space of A is the range of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.
 d. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then Col A is $\mathbb{R}^m$.
 e. The kernel of a linear transformation is a vector space.
 f. Col A is the set of all vectors that can be written as $A\mathbf{x}$ for some $\mathbf{x}$.

26. a. A null space is a vector space.
 b. The column space of an $m \times n$ matrix is in $\mathbb{R}^m$.
 c. Col A is the set of all solutions of $A\mathbf{x} = \mathbf{b}$.
 d. Nul A is the kernel of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.
 e. The range of a linear transformation is a vector space.
 f. The set of all solutions of a homogeneous linear differential equation is the kernel of a linear transformation.

27. It can be shown that a solution of the system below is $x_1 = 3$, $x_2 = 2$, and $x_3 = -1$. Use this fact and the theory from this section to explain why another solution is $x_1 = 30$, $x_2 = 20$, and $x_3 = -10$. (Observe how the solutions are related, but make no other calculations.)

$$\begin{aligned} x_1 - 3x_2 - 3x_3 &= 0 \\ -2x_1 + 4x_2 + 2x_3 &= 0 \\ -x_1 + 5x_2 + 7x_3 &= 0 \end{aligned}$$

28. Consider the following two systems of equations:

$$\begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 1 \\ 4x_1 + x_2 - 6x_3 &= 9 \end{aligned} \qquad \begin{aligned} 5x_1 + x_2 - 3x_3 &= 0 \\ -9x_1 + 2x_2 + 5x_3 &= 5 \\ 4x_1 + x_2 - 6x_3 &= 45 \end{aligned}$$

It can be shown that the first system has a solution. Use this fact and the theory from this section to explain why the second system must also have a solution. (Make no row operations.)

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 a. Show
 b. Let B
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 c. Show
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 d. Descr

34. (*Calculus*
For $\mathbf{f}$ in
that $\mathbf{F}(0)$
describe
Section 4

the spanning property.

$$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix}\right\} \qquad \left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}4\\5\\6\end{bmatrix},\begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$$

Linearly independent but does not span $\mathbb{R}^3$ — A basis for $\mathbb{R}^3$ — Spans $\mathbb{R}^3$ but is linearly dependent ■

PRACTICE PROBLEMS

1. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}-2\\7\\-9\end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathbb{R}^3$. Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for $\mathbb{R}^2$?

2. Let $\mathbf{v}_1 = \begin{bmatrix}1\\-3\\4\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}6\\2\\-1\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}2\\-2\\3\end{bmatrix}$, and $\mathbf{v}_4 = \begin{bmatrix}-4\\-8\\9\end{bmatrix}$. Find a basis for the subspace W spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

3. Let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $H = \left\{\begin{bmatrix}s\\s\\0\end{bmatrix} : s \text{ in } \mathbb{R}\right\}$. Then every vector in H is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ because

$$\begin{bmatrix}s\\s\\0\end{bmatrix} = s\begin{bmatrix}1\\0\\0\end{bmatrix} + s\begin{bmatrix}0\\1\\0\end{bmatrix}$$

SG Mastering: Basis 4-9

Is $\{\mathbf{v}_1, \mathbf{v}_2\}$ a basis for H?

4.3 EXERCISES

Determine whether the sets in Exercises 1–8 are bases for $\mathbb{R}^3$. Of the sets that are *not* bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

1. $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$

2. $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}$

3. $\begin{bmatrix}1\\0\\-3\end{bmatrix}, \begin{bmatrix}3\\1\\-4\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\end{bmatrix}$

4. $\begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}2\\-3\\2\end{bmatrix}, \begin{bmatrix}-8\\5\\4\end{bmatrix}$

5. $\begin{bmatrix}3\\-3\\0\end{bmatrix}, \begin{bmatrix}-3\\7\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\5\end{bmatrix}$

6. $\begin{bmatrix}1\\2\\-4\end{bmatrix}, \begin{bmatrix}-4\\3\\6\end{bmatrix}$

7. $\begin{bmatrix}-2\\3\\0\end{bmatrix}, \begin{bmatrix}6\\-1\\5\end{bmatrix}$

8. $\begin{bmatrix}1\\-2\\3\end{bmatrix}, \begin{bmatrix}0\\3\\-1\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}, \begin{bmatrix}0\\0\\-1\end{bmatrix}$

Find bases for the null spaces of the matrices given in Exercises 9 and 10. Refer to the remarks that follow Example 3 in Section 4.2.

9. $\begin{bmatrix}1 & 0 & -2 & -2\\0 & 1 & 1 & 4\\3 & -1 & -7 & 3\end{bmatrix}$

10. $\begin{bmatrix}1 & 1 & -2 & 1 & 5\\0 & 1 & 0 & -1 & -2\\0 & 0 & -8 & 0 & 16\end{bmatrix}$

11. Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + 2z = 0$. [*Hint:* Think of the equation as a "system" of homogeneous equations.]

12. Find a basis for the set of vectors in $\mathbb{R}^2$ on the line $y = -3x$.

In Exercises 13 and 14, assume that A is row equivalent to B. Find bases for Nul A and Col A.

13. $A = \begin{bmatrix}-2 & 4 & -2 & -4\\2 & -6 & -3 & 1\\-3 & 8 & 2 & -3\end{bmatrix}$, $B = \begin{bmatrix}1 & 0 & 6 & 5\\0 & 2 & 5 & 3\\0 & 0 & 0 & 0\end{bmatrix}$

4.3

14. $A = \begin{bmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

In Exercises 15–18, find a basis for the space spanned by the given vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

15. $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -8 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 10 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -6 \\ 9 \end{bmatrix}$

16. $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

17. [M] $\begin{bmatrix} 2 \\ 0 \\ -4 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \\ 8 \\ -3 \\ 15 \end{bmatrix}, \begin{bmatrix} -8 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

18. [M] $\begin{bmatrix} -3 \\ 2 \\ 6 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -9 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -4 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -14 \\ 0 \\ 13 \end{bmatrix}, \begin{bmatrix} -6 \\ 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

19. Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, and also let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It can be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for H. There is more than one answer.

20. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 4 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 5 \\ -6 \\ -14 \end{bmatrix}$. It can be verified that $2\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

In Exercises 21 and 22, mark each statement True or False. Justify each answer.

21. a. A single vector by itself is linearly dependent.

b. If $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$, then $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for H.

c. The columns of an invertible $n \times n$ matrix form a basis for $\mathbb{R}^n$.

d. A basis is a spanning set that is as large as possible.

e. In some cases, the linear dependence relations among the columns of a matrix can be affected by certain elementary row operations on the matrix.

22. a. A linearly independent set in a subspace H is a basis for H.

b. If a finite set S of nonzero vectors spans a vector space V, then some subset of S is a basis for V.

c. A basis is a linearly independent set that is as large as possible.

d. The standard method for producing a spanning set for Nul A, described in Section 4.2, sometimes fails to produce a basis for Nul A.

e. If B is an echelon form of a matrix A, then the pivot columns of B form a basis for Col A.

23. Suppose $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$. Explain why $\{\mathbf{v}_1, \ldots, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

24. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a linearly independent set in $\mathbb{R}^n$. Explain why $\mathcal{B}$ must be a basis for $\mathbb{R}^n$.

25. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and let H be the set of vectors in $\mathbb{R}^3$ whose second and third entries are equal. Then every vector in H has a unique expansion as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, because

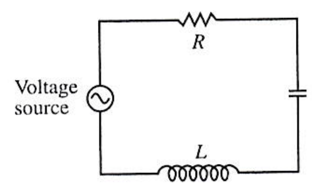
$$\begin{bmatrix} s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + (t - s)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for any s and t. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for H? Why or why not?

26. In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

27. Let V be the vector space of functions that describe the vibration of a mass–spring system. (Refer to Exercise 19 in Section 4.1.) Find a basis for V.

28. (*RLC circuit*) The circuit in the figure consists of a resistor (R ohms), an inductor (L henrys), a capacitor (C farads), and an initial voltage source. Let $b = R/(2L)$, and suppose R, L, and C have been selected so that b also equals $1/\sqrt{LC}$. (This is done, for instance, when the circuit is used in a voltmeter.) Let $v(t)$ be the voltage (in volts) at time t, measured across the capacitor. It can be shown that v is in the null space H of the linear transformation that maps $v(t)$ into $Lv''(t) + Rv'(t) + (1/C)v(t)$, and H consists of all functions of the form $v(t) = e^{-bt}(c_1 + c_2 t)$. Find a basis for H.



Exercises 29 and 30 show that every basis for $\mathbb{R}^n$ must contain exactly n vectors.

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